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Scalar-Tensor Theory and Propagating Torsion Theory

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We prove that scalar conformal transformations can convert the variational principle of the propagating torsion theory into the variational principle of general scalar-tensor theory, and show that scalar-tensor theory is conformally equivalent to propagating torsion theory.

Harrison (1972) has proved that scalar-conformal transformations not only interconvert the various versions of the scalar-tensor theory [except the scalar-tensor theory with short-range term, such as the models suggested by O'Hanlon (1972) and Xu (1989*a*,*b*)], but also convert the scalar-tensor variational principle into the variational principle of general relativity. It is the purpose of this brief report to further show that scalar-conformal transformations can convert the variational principle of the propagation torsion theory into the variational principle of general scalar-tensor theory which is given by Harrison. This means that scalar-tensor theory may be embedded in the propagating torsion theory.

In the Riemann–Cartan spacetime U_4, the connection is

$$\Gamma^k_{ij} = \begin{Bmatrix} k \\ ij \end{Bmatrix} - K_{ij}{}^k \tag{1}$$

where $\{^{\,k}_{ij}\}$ is the Christoffel symbol and

$$K_{ij}{}^k = -S_{ij}{}^k + S^k{}_{ij} + S^k{}_{ji} \tag{2}$$

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is the contorsion tensor, and

$$S_{ij}{}^{k}=\tfrac{1}{2}(\Gamma^{k}_{ij}-\Gamma^{k}_{ji}) \tag{3}$$

is the torsion tensor. The Ricci tensor is

$$\begin{aligned}R_{ij}=R_{ij}(\{\cdot\})-\frac{\partial}{\partial x^{l}}K_{ij}{}^{l}+\frac{\partial}{\partial x^{j}}K_{il}{}^{l}-\begin{Bmatrix}k\\ ij\end{Bmatrix}K_{kl}{}^{l}-K_{ij}{}^{l}\begin{Bmatrix}k\\ lk\end{Bmatrix}\\ +K_{ij}{}^{k}K_{kl}{}^{l}+\begin{Bmatrix}k\\ il\end{Bmatrix}K_{kj}{}^{l}+K_{il}{}^{k}\begin{Bmatrix}l\\ kj\end{Bmatrix}-K_{il}{}^{k}K_{kj}{}^{l}\end{aligned} \tag{4}$$

where $R_{ij}(\{\cdot\})$ is the Ricci tensor in Riemann spacetime V_4, namely, the Ricci tensor with respect to the Christoffel symbol. There exist two identical equations in the Riemann–Cartan spacetime U_4 as follows:

$$G^{k}{}_{j;k}-2G^{k}{}_{j}S_{kl}{}^{l}=2G^{k}{}_{l}S_{kj}{}^{l}+R^{ik}{}_{lj}T_{ik}{}^{l} \tag{5}$$

$$2T_{ij}{}^{l}{}_{;l}=G_{ij}-G_{ji}+4S_{lk}{}^{k}S_{ij}{}^{l} \tag{6}$$

where

$$T_{ij}{}^{k}=S_{ij}{}^{k}+\delta^{k}_{i}S_{jl}{}^{l}-\delta^{k}_{j}S_{il}{}^{l} \tag{7}$$

is the modified torsion tensor and

$$G^{i}{}_{j}=R^{i}{}_{j}-\tfrac{1}{2}\delta^{i}_{j}R \tag{8}$$

is the Einstein tensor. The semicolon ";" denotes covariant derivative with respect to the connection Γ^{k}_{ij}.

We take the torsion tensor as follows:

$$S_{ij}{}^{k}=\tfrac{1}{2}b(\delta^{k}_{i}\Lambda_{,j}-\delta^{k}_{j}\Lambda_{,i}) \tag{9}$$

where $\Lambda=\ln\lambda$, λ is the scalar function, and Λ is called the torsional potential. The comma "," indicates the usual derivative. b is a parameter which is independent of the spacetime point. Substituting equation (9) into equation (6), we obtain the symmetry as follows:

$$G_{ij}=G_{ji},\qquad R_{ij}=R_{ji} \tag{10}$$

Substituting equation (9) into equation (5) and using the symmetry (10), we obtain the identical equation

$$G^{li}{}_{|l}=2bR^{il}\Lambda_{,l} \tag{11}$$

where the vertical bar symbol "|" denotes the covariant derivative using only the Christoffel symbols of the metric. Substituting equation (9) into equation (4), we get

$$R_{ij}=R_{ij}(\{\cdot\})+2b\Lambda_{,i|j}+bg_{ij}\Lambda^{,k}{}_{|k}+2b^2(\Lambda_{,i}\Lambda_{,j}-g_{ij}\Lambda^{,k}\Lambda_{,k}) \tag{12}$$

We thus have the curvature scalar

$$R=R(\{\cdot\})-6b^2\Lambda^{,k}\Lambda_{,k}+6b\Lambda^{,k}{}_{|k} \tag{13}$$

and the Einstein tensor

$$G_{ij}=G_{ij}(\{\cdot\})+2b\Lambda_{,i|j}-2bg_{ij}\Lambda^{,k}{}_{|k}+b^2(2\Lambda_{,i}\Lambda_{,j}+g_{ij}\Lambda^{,k}\Lambda_{,k}) \tag{14}$$

where $R(\{\cdot\})$ and $G_{ij}(\{\cdot\})$ are the curvature scalar and the Einstein tensor, respectively, with respect to the Christoffel symbol. Substituting equations (12) and (14) into equation (11), we obtain the identical equation

$$G^{ik}(\{\cdot\})_{|k}=0 \tag{15}$$

The variational principle of the new propagating torsion theory is

$$\begin{aligned}\delta S&=\delta\int(R+kL)\sqrt{-g}\,d^4x\\&=\delta\int[R(\{\cdot\})-6b^2\Lambda^{,k}\Lambda_{,k}+kL]\sqrt{-g}\,d^4x\end{aligned} \tag{16}$$

where L is the matter-Lagrangian density. k is a coupling constant. Here we have discarded the divergence term in equation (13).

Applying in succession to equation (16) the scalar-conformal transformations

$$\lambda=\varphi^s \tag{17}$$

$$g_{ij}=\varphi^t\bar{g}_{ij} \tag{18}$$

and also using

$$L(g_{ij})=\varphi^{-r}\bar{L}(\bar{g}_{ij}) \tag{19}$$

after discarding a divergence term, we can write equation (16) as

$$\delta S=\delta\int[\varphi^t\bar{R}(\{\cdot\})+\mu\varphi^{t-2}\varphi^{,k}\varphi_{,k}+k\varphi^B\bar{L}](-\bar{g})^{1/2}\,d^4x=0 \tag{20}$$

in which φ is the scalar function; s, t, and r are constants; and

$$\begin{aligned}\mu&=3(\tfrac{1}{2}t^2-2b^2s^2)\\B&=2t-r\end{aligned} \tag{21}$$

Applying equation (18) to equation (15), we get that

$$\bar{G}^{ik}(\{\cdot\})_{|k}=0 \tag{22}$$

Independent variation in $\bar{g}_{ij}$ and φ of equation (20) gives the field equations

$$\bar{R}_{ij}(\{\cdot\})-\tfrac{1}{2}\bar{g}_{ij}\bar{R}(\{\cdot\})-t\varphi^{-1}(\varphi_{,i|j}-\bar{g}_{ij}\Box\varphi)-(t^2-t-\mu)\varphi^{-2}\varphi_{,i}\varphi_{,j}$$
$$+(t^2-t-\tfrac{1}{2}\mu)\bar{g}_{ij}\varphi^{-2}\varphi^{,l}\varphi_{,l}-\tfrac{1}{2}k\varphi^{B-t}\bar{T}_{ij}=0 \tag{23}$$

$$t\bar{R}(\{\cdot\})-\mu(t-2)\varphi^{-2}\varphi^{,k}\varphi_{,k}-2\mu\varphi^{-1}\Box\varphi+kB\varphi^{B-t}\bar{L}=0 \tag{24}$$

where

$$\bar{T}_{ij}=-\frac{1}{(-\bar{g})^{1/2}}\frac{\partial((-\bar{g})\bar{L})}{\partial\bar{g}^{ij}} \tag{25}$$

is the energy-momentum tensor of matter. Substituting equations (23) and (24) into equation (22), we obtain the following equation:

$$(t-1)\left[\varphi_{,j}\bar{R}^{j}{}_{l}(\{\cdot\})+\frac{\mu}{t}\varphi^{-2}\varphi_{,l}\varphi^{,k}\varphi_{,k}-t\varphi^{-1}\varphi^{,j}\varphi_{,l|j}+\left(t-\frac{\mu}{t}\right)\varphi^{-1}\varphi_{,l}\Box\varphi\right.$$
$$\left.-\tfrac{1}{2}k\varphi^{B-t}\varphi_{,j}\bar{T}_{l}{}^{j}\right]+\frac{B}{2}k\varphi^{B-t}\left(\varphi_{,j}\bar{T}_{l}{}^{j}-\frac{1}{t}\varphi_{,l}\bar{L}\right)$$
$$+\tfrac{1}{2}k\varphi^{B-t+1}\bar{T}_{l}{}^{j}{}_{|j}=0 \tag{26}$$

Equations (20)–(26) are of the same forms as Harrison's results. If $t=1$, $s=1$, $r=2$, and $\mu=-\omega$, then $B=0$, $b^2=(2\omega+3)/12$, and equations (20)–(26) reduce to the Brans–Dicke (1961) theory. Therefore, scalar-tensor theory is conformally equivalent to the propagating torsion theory. The scalar field plays the torsional potential role. The torsion can propagate in vacuum.

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